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To:Doig, Margaret <MargaretDoig@creighton.edu>

Dear Margaret,

It is my pleasure to inform you that your revised article, “Condorcet in Math Class,” has been accepted for publication in the next volume of *XVIII New Perspectives on the Eighteenth Century* 20:1 (2023), http://www.seasecs.org/npec. The 2023 journal will go into production in May 2023, be printed in June 2023, and mailed out in July 2023.

Thank you for submitting the revisions based on the second round of readers’ comments. Please know that there may be some final edits once I begin formatting the essay for the volume, which I will ask you to turn around quickly.

Sincerely yours,
Joe

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"Célébrons nos différents accents au lieu de mettre l'accent sur nos différences" -Boucar Diouf

Condorcet in Math Class:
How an Eighteenth Century Philosophe Enriches the Modern Undergraduate Experience

Margaret I. Doig, Creighton University

Marie-Jean-Antoine-Nicolas de Caritat, marquis de Condorcet (1743-1794), one of the youngest and last of the *philosophes*, is perhaps most widely read today for his radical political opinions on economic and religious freedom, educational reform, and equal treatment under the law regardless of race or gender. During his lifetime, though, his political writings were the source of some contention,[1] while it was his mathematical papers that were generally respected (if perhaps infrequently read outside a small community of specialists). In addition to some work on calculus for which he was admitted to the French Royal Academy of Sciences, he contributed significantly to the early development of probability theory and was one of the first to apply mathematical techniques in the social sciences. In the course of developing these applications, Condorcet contributed several key ideas to the study of voting, and these ideas today play a significant role in many undergraduate mathematics and social science courses.

Voting theory, more formally included under the umbrella term "social choice theory," is the study of collective decision-making. It mathematically studies how to merge the preferences of each member of a group into a communal judgment in a way that maximizes fairness, or the adherence of the collective will to the individual.

We frequently cover this topic in liberal arts mathematics courses, and several of our key ideas flow from Condorcet. He did not originate voting theory, nor was he the only one to advocate its main mathematical ideas; however, he was primary in the breadth of his investigation and in his treatment of its logical underpinnings, and his presentation pioneered the form our discussion takes in the classroom. Condorcet's work has clear and significant implications for our undergraduates' educational experience.

On the educational context

My colleagues and I routinely teach voting theory as part of an introductory course for humanities students and others not needing calculus. The course is designed to teach basic quantitative literacy, develop analytical and logical skills, and help students incorporate quantitative techniques and language into their intellectual life. There is also invariably an element of combating math anxiety and preparing these students for a future where they can meet a mathematician at a party and refrain from saying, "Oh, I hate math, and it hates me."

There are numerous texts available for introductory mathematics courses, and many contain chapters on voting.[2] Voting theory is highly pertinent to such a course because of its relevance to modern political life. These texts explicitly cover the technical descriptions of various voting methods in use or under consideration around the world today, along with properties they obey and some paradoxical situations where the outcomes seem startlingly unfair. Voting theory also contributes to the students' mathematical development by fostering their logic skills. They must examine a precise set of definitions and algorithms to determine which outcomes are mathematically possible, and they must check particular voting methods against a rigorous set of formal conditional statements about what "ought" to happen in a fair system.

We see Condorcet's name regularly in these texts. He was the first to extensively discuss structural criteria for an election to be fair and paradoxical situations where it might not be.

Some of the formal language we use to phrase these criteria comes from later mathematicians, but we roughly follow Condorcet's train of thought and the spirit of his arguments because they are so accessible to students. Janet Heine Barnett has in fact proposed teaching this entire unit directly through the lens of the French Enlightenment thinkers, accompanied by excerpts from their original works.[3] In addition to covering the technical content in an engaging way, these materials use the Enlightenment development of voting theory as a demonstration of how new mathematics is created and how it fits into and derives meaning from its social context.

In spring 2022, I also taught an interdisciplinary mathematics and political science course on voting theory for our undergraduate honors program, a so-called "sources and methods" course. The class included almost twenty high-achieving students spread across the arts and sciences, ranging from a few political science majors trying to knock out their math requirement with a minimum of "abstract nonsense," to a few math majors trying to knock out their social science requirement with a minimum of "endless talking," and everyone in between. We covered the technical aspects of social choice theory, although much more rigorously than in the introductory course, and we explicitly studied voting procedures as logical structures subject to analytical investigation.[4] Just as Barnett proposed (although using additional primary and secondary materials), we watched the dramatic history of social choice theory unfold and saw how it has impacted the society which birthed it.

On the historical context

To understand Condorcet's impact in the classroom, we must understand how his work fits into the historical development of voting theory.

In the words of Iain McLean, social choice theory has been discovered four times and lost three.[5] The first[6] formal consideration of a voting system as a logical structure comes from Ramon Llull, a thirteenth-century Catalan philosopher and apologist who learned combinatorics (probably) from Muslim mathematicians and applied it to a number of humanistic subjects, including ecclesiastical elections.[7] Two centuries later, Nicolas of Cusa, also known as Cusanus, read some of Llull's work and wrote similarly on electing the Holy Roman Emperor.[8] Between them, they formally proposed methods of voting which anticipated both Condorcet and his contemporary Jean-Charles de Borda.[9] Neither used modern mathematical language, nor did they give mathematical arguments for how well their voting procedures converted the opinions of individuals into a communal choice, but they did mention psychological and social concerns about fairness, including the freedom of the voter from undue influence, the makeup of the electorate, and the equal or unequal weighting of individual voters. Their work was lost until the late twentieth century, perhaps because ad hoc methods sufficed for the occasional voting scenario which arose in the late Middle Ages.

Social choice theory next appeared during the Enlightenment. The Scientific Revolution necessitated the rebirth of mathematics as a mechanism for describing and explaining the patterns observed in the natural world, such as the development of calculus to support physics. In the midst of this movement, Condorcet produced *Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix (An essay on the application of probability theory to plurality decision-making,* 1785). This 495-page treatise formalized the fledgling field of probability theory and then used it to analyze the outcomes of collective decision-making. Condorcet considered a situation where some body (assembly of voters, legislature, jury, etc.) must come together and combine individual preferences to make a communal decision. He initially approached it in terms of quantifying the probability of various factors (e.g., that individuals would err) and calculating the likelihood that the group would arrive at the "correct" outcome.

In the course of his probabilistic study, though, Condorcet examined what a correct outcome could be, that is, which possible outcomes would reasonably reflect the particular makeup of voters' preferences.[10] He first considered the logical structure of the conventional (plurality) method of voting and identified a situation where he considered the winner to be unfair. He provided an alternate system, a "pairwise comparison procedure," and a logical argument for why it was more representative of the individuals' preferences than the alternatives. We now formalize this type of analysis with what we call the "fairness criteria" in class, and we analyze voting systems to see if they satisfy them. Condorcet's primary principle we call the "Condorcet criterion"; we find another criterion mentioned in passing in the 1785 *Essai* as well as a third in a later work, *Essai sur la constitution et les fonctions des assemblées provinciales* (*On the constitution and the functions of provincial assemblies,* 1788), where he criticized Borda's method.[11]

A week after Condorcet announced his treatise, Borda, a military officer and applied mathematician, read a nine-page paper of his own to the Royal Academy of Sciences on the subject of electing new members to the Academy.[12] He criticized the conventional method for the same reasons as Condorcet and proposed an alternate method that assigned points to candidates based on the voters' preferences, which is an example of what we now call a "Borda count" or "scoring procedure." While Borda's system was actually adopted by the Academy in 1795 for electing new members, Condorcet's proposal seems to have languished, perhaps because it would have been prohibitively complicated to implement.

In the late nineteenth century, social choice was revisited by British mathematician Charles Dodgson, the pseudonymous Lewis Carroll, who appears not to have read Borda or Condorcet.[13] He alternately proposed a voting system almost identical to Borda's, a system very similar to Condorcet's, and finally a hybrid procedure, justified in part by a mathematical analysis of fairness very similar in flavor to Condorcet's.[14] Electoral reform become a hot topic around the world at this time, and a number of other systems were discussed. A category of systems involving transferable votes (including what we teach as "Instant runoff voting") was first considered (and rejected) by Condorcet, reintroduced with some refinement of implementation by architecture professor W. R. Ware, publicized by British scholar Sir Thomas Hare in the context of an expanded system he proposed for Great Britain, and mathematically investigated by E. J. Nanson.[15] The very modern language we use in class to formulize the principles of fairness owes much to Dodgson and his contemporaries.

The final and most formal chapter of social choice theory started in the mid-twentieth century, most prominently with Kenneth Arrow, who shared a Nobel Prize in economics for this work. Writing in ignorance of those who came before,[16] Arrow considered possible voting methods as a set of abstract mathematical objects, formulated his own criteria (axioms, in his words), and asked which methods satisfy which criteria. Arrow also speculated on the applicability of this work to evaluating the impact of social alternatives outside the realm of voting.[17] This may be seen as the most formal and recognizably mathematical formulation of voting theory, and it has undoubtedly influenced the fact that we use voting theory to teach logic; that said, the language and method of argument are too abstruse to use in our introductory course, and they appear in the honors course only as a brief example of the methods of modern social choice theory.

On the value of voting theory in the mathematical curriculum

Voting theory serves several distinct learning objectives: understanding the manifold ways the same information may be fruitfully presented and the value judgments needed to select one appropriate to the context; thinking and speaking precisely and correctly during the analytical process; discussing a concept of fairness which is rooted in non-mathematical experience and worldview but using formal mathematical language and tools; and learning the principles of logic. Of course, we also hope our students will emerge as more thoughtful and educated voters themselves.

On methods of presenting information

Students are often surprised when they first encounter voting theory since it does not look like math, and it does not feel like math. They expect formulas and numbers, not lists of rules and conditional statements. This is a prime opportunity to discuss the many different ways in which we can methodically prepare information for analysis.

A standard mathematics textbook will distinguish among verbal, numeric, visual, and algebraic (or symbolic) presentations of data. The same information may be presented in several of these ways, although some are usually superior, and intuition, common sense, and a consideration of the desired audience must guide the student's selection of format. For example, we define voting systems for our students primarily using verbal descriptions or sets of rules (as we also do in this article), while the modern social choice literature usually defines them symbolically using the notation of set theory and algebra. As my honors class learned when we read an excerpt from Arrow,[18] a voting system may be defined as a ranking function; in a system named *R,* we write *aRb* to mean candidate *a* ranks at least as high as candidate *b.* Arrow assigns to the system many of the properties we would wish for it, which he phrases in the form of axioms. For example, given any two candidates, we may compare them:

$$Axiom\ I: For\ all\ a\ and\ b, aRb\ or\ bRa.$$

Similarly, ranking is "transitive," meaning that the relative rankings of pairs of candidates must be compatible with one another:

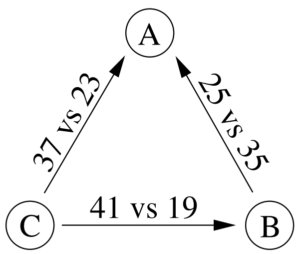
$$Axiom\ II: For\ all\ a, b, and\ c, aRb\ and\ bRc\ together\ imply\ aRc.$$

If we are examining a particular voting scenario, we could also describe it verbally, or we could list a table of all the voters and their individual preferences. Condorcet and Borda did both. Today, though, to save space and sanity, we usually consolidate voters with identical preferences to form the "preference schedule," as we can for an example from Condorcet with three candidates and 60 voters:[19]

	23 voters	19 voters	16 voters	2 voters
1st place	A	B	C	C
2nd place	C	C	B	A
3rd place	B	A	A	B

It is in having students reduce a complex real-life situation to a small table that we encounter one of the major difficulties of mathematics. The abstraction which facilitates analysis of some information does so by removing other information, and this inherently involves making value judgments and simplifications which may impair the applicability of our conclusions. For example, we assume that all voters are rational, and that they rank candidates in a compatible (transitive) fashion. We assume they can rank all candidates with equal confidence and will not change their minds if presented with a ballot in a different format (for example, being asked to give a single ranking of all candidates together versus being asked to compare each pair of candidates).

A third distinct method of presenting information arises when we discuss Condorcet's ideas. To try to convert all the individual voters' opinions into one coherent group opinion, he compared the candidates pairwise to assess their relative strengths; for example, if the voters above were required to choose between only candidates A and C, 23 of them (the first column) would select A, while the remaining 37 (from the second, third, and fourth columns) would select C. Similarly, between A and B, A would be favored by 25 voters, while B would be favored by 35, and, between B and C, B would receive 19 and C 41. We could visualize this by drawing the candidates with an arrow between each pair, from the more preferred candidate to the lesser, labelled by the number of voters preferring each, or in a table listing the ratios:



	... vs A	... vs B	... vs C
A ...	N/A	25/35	23/37
B ...	35/25	N/A	19/41
C ...	37/23	41/19	N/A

Today, we call this set of pairwise comparisons a "Condorcet tournament," and we usually present it using this graph or table. While Condorcet followed the practices of his day and included no figures in his work, modern students and mathematicians alike tend towards visualizations, and this is a natural place for one to arise. Many students in fact produce something like these without any prompting.

As we work through the material, we have many opportunities to discuss the respective merits of different formats for presenting mathematical information. We can ask students which they prefer in a given situation, and their answers vary greatly based on context but are surprisingly consistent as well as usually well justified. For example, if we ask, "How do we analyze these two candidates competing for a seat in the legislature?" their answer will probably be supported by a preference schedule, whereas "How do we rank these four basketball teams in the semifinals?" will often inspire a graph of the Condorcet tournament. In each of these cases, math involves an act of judgment: we consider factors like ease of communication and simplicity of calculation in light of the situational forces behind the election.

On the use of precise definitions and algorithms

Voting theory also requires great attention to detail. It introduces students to mathematically precise language, the difficulties of formulating it, and the problems that arise from failing to do so well. Understanding a voting system requires engaging with a fairly familiar topic but in perhaps unexpectedly precise terms: for example, if asked to define the familiar “plurality system,” students may say that voters vote for candidates, and the one with the most votes wins. It is not uncommon for students to forget to mention that each voter selects only one candidate, or that this candidate should be their top choice, yet these are essential details. Similarly, most students immediately name the plurality system as the one in use for election to the U.S. House and Senate, but this is actually incorrect. First, the plurality system as strictly defined does not address ties, whereas every state has added some method to do so; second, as Georgia famously demonstrated in 2020, several states actually use a “majority system” where the winner must receive more than 50% of the vote.

Condorcet and the other social choice theorists themselves faced the same challenge our students do. They had to carefully and thoroughly explain the details of each system and the motivation behind each design decision, whether it came from societal context, the practicalities of voting, or a mathematical analysis of how to convert the voters’ opinions into a consensus.

One of the simpler systems to define is the “Borda count,” an example of a “scoring procedure,” variously derived by Llull, Cusanus, Borda, and Dodgson.[20] Such a system is motivated by a concern about whether the simple plurality winner can adequately reflect the intent of the population. As Borda phrased it, after giving an example similar to Condorcet’s above, where candidate A is ranked last by more than half the voters yet still ranked first by more voters than any other candidate, “In the collective opinion of the voters, then, candidate A is decidedly inferior to both B and C ... It follows that the will of the voters demands that candidate A be excluded. However, if we use the conventional election method, such a candidate might actually obtain the plurality.”[21] Instead of the plurality system, Borda proposes, we should determine how all voters rank all candidates and assign points based on these rankings, selecting the candidate(s) with the most points as the winner(s). Using Condorcet’s example, 23 voters ranked A first, so we assign it 2 points for each of these voters; 2 ranked it second, so we assign 1 point for each of these; and 35 ranked it last, so we give no points for these, totaling 48 points. Likewise, B receives 54 and C 78, rendering C the winner (with B in second and A in third, if we wish to rank all three candidates). Our students find this type system intuitively reasonable as they have all seen point-based systems in use outside politics, but it is still a challenge for them to formulate the system precisely, to describe the method of assigning and totaling points (e.g., it is easy forget that each column in the preference schedule assigns points *weighted by the number of voters* represented by the column), to explicitly state the point values in use (which vary with the number of candidates), and to acknowledge that a different choice of point values may also have merit in a given situation (e.g., allotting a first choice candidate twice as many points as a second).

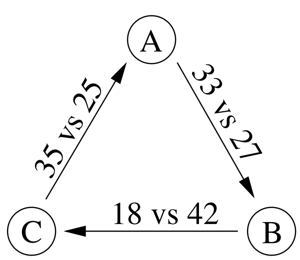
Another alternative to the plurality system is the family of transferable voting systems, including “instant-runoff voting,” currently in use in Maine and Alaska for some elections where it is usually called “ranked-choice voting.” A simple system was first considered by Condorcet, and more sophisticated versions were proposed by Ware, publicized by Hare, and mathematically analyzed by Nanson.[22] Instant-runoff voting works like an inverted plurality vote. Each voter ranks all the candidates, and we simulate a series of runoffs where losing candidates are eliminated one by one on the basis of a plurality election until the winner or winners remain. In Condorcet’s example, an initial plurality vote would result in 23 votes for A, 19 for B, and 18 for C, thus ranking the candidates A, then B, then C. Instead of declaring the winner immediately, we declare C to be the loser and consider how this election would have proceeded without it: 16 of C’s voters would have gone for B and 2 for A, which yields 25 votes for A and 35 for B, so we now declare A to be the loser, eliminate it, and are left with B as the winner, A second, and C third. Once more, our students initially find the system appealing as a run-off system, but they must describe the algorithm without being misled by the term "run-off" (there is no return to the polls), and they must, most importantly, remember that eliminating one candidate results in an immediate transfer of votes from that candidate's supporters to other candidates (in other words, instant-runoff voting requires the entire preference schedule and not merely the first row).

Finally, we have the "pairwise comparison procedures," variously invented by Llull, Condorcet, Dodgson, and finally Arthur Copeland in the twentieth century.[23] We begin with a Condorcet tournament comparing each pair of candidates. Llull and Copeland analyzed the tournament by counting how many pairwise comparisons each candidate won (with a minor difference in the treatment of ties) and declaring the candidate(s) with the most to be the winner(s). Dodgson had a more strenuous condition: if any one candidate is strong enough to have won *every* comparison as C did in the example above, what we now call a "Condorcet winning candidate," then that candidate is declared the winner. This condition is not overly restrictive: for example, if there is a candidate who could have won a majority vote, then that candidate is a Condorcet winning candidate, though not vice versa. Condorcet, on the other hand, attempted to convert the tournament directly to a full ranking of all the candidates. That is, C beats A and B, and B beats A, so we rank the candidates C, then B, then A, and declare C the winner.

All four pairwise comparison systems produce the same winner if there is a Condorcet winning candidate. Alas, there often isn't one, as Condorcet himself demonstrated with this example:[24]

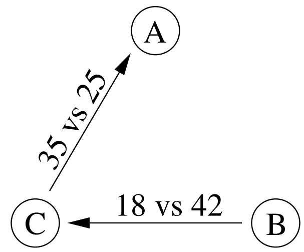
	23 voters	17 voters	2 voters	10 voters	8 voters
1st place	A	B	B	C	C
2nd place	B	C	A	A	B
3rd place	C	A	C	B	A

Here, A beats B, B beats C, and C beats A, and we have a conundrum we now call "Condorcet's paradox" or "a Condorcet cycle."



	... vs A	... vs B	... vs C
A ...	N/A	33/27	25/35
B ...	27/33	N/A	42/18
C ...	35/25	18/42	N/A

Such a cycle prevents a full ranking of the candidates, and, if it occurs among the top candidates, then it also obstructs the presence of a Condorcet winning candidate. Depending on the situation, Copeland's and Llull's methods might declare all three of these candidates tied as winner, although they might be able to distinguish between them on the basis of comparisons to other candidates.[25] To the contrary, Dodgson declared this pairwise comparison method to be inconclusive in the presence of a Condorcet cycle; in fact, in his 1876 pamphlet, he proposed this as the second of a series of elections, preceded by a majority vote and, if needed, succeeded by a Borda count. Condorcet likewise saw some ambiguity in this situation: "We should point out, finally, that these contradictory [pairwise comparisons] cannot occur unless there is some uncertainty in opinions."[26] With his probabilistic point of view, he concluded that these three pairwise comparisons should not be given equal weight. The voters preferred B to C by a ratio of 42:18, whereas they preferred C to A by 35:25, and A to B by only 33:27, so A vs. B was the least decisive of the three comparisons, or the situation which was most likely to have been affected by any weaknesses in the electoral methods or errors in the voters' judgments. Condorcet eliminated this comparison completely and proceeded on the basis of the other two:



In this case, B beats C who beats A, so B is declared the winner.

When we teach this topic, we begin by introducing a graph of the Condorcet tournament, and many students immediately convert it into a ranking of the candidates, thus deriving the simple version of Condorcet's method on their own. When we introduce an example with a Condorcet cycle, however, their progress stalls and their confidence drops, so we teach them the resolution Copeland derived and treat his method as our official example of this family of voting procedures. Copeland's method is challenging enough for the students to keep straight from the other methods: as with instant-runoff, it can be difficult for them to remember that a pair of candidates must be treated truly in isolation (i.e., they must use the entire preference schedule and not just the first row), to record the outcomes about the pairwise comparisons in an effective and legible fashion (whether in a list, a table, or a graph), to make sure all pairs of candidates are considered, and, finally, to understand that the process of visually checking the graph for a Condorcet cycle is distinct from the process of counting comparisons to calculate a winner under Copeland's method.

On fairness

Implicit in our careful definitions of voting systems are the motivations that lay behind the design of each. Llull and Cusanus offered primarily societal justifications. Borda offered the hint of a mathematical justification when he argued that, if more than half the voters were to rank a candidate last, then it would be the "collective opinion of the voters" that the candidate is inferior and "the will of voters" would demand the candidate be excluded.[27] Condorcet gave the same criticism and also discussed the complementary idea, that, if some candidate were ranked first by the majority of the voters, then that candidate ought to win. For example, when discussing possible error in the voters' judgment, he wrote, "For if more than half of the voters admit an unworthy candidate, it is evident that they would have elected him under any form of election."[28] We call this a "majority winning candidate" and the principle that such a candidate ought to be elected the "majority criterion,"[29] and we teach it to our students as a logical condition used to assess whether a voting system is "fair," that is, whether the results reasonably reflect the will of the voters. The plurality and the pairwise comparison systems satisfy it, although, surprisingly, the Borda count does not.[30] Our students readily agree with both Borda and Condorcet, and many actually verbalize this without prompting if shown examples similar to Borda's and Condorcet's.

Condorcet's criticism of the plurality system went much farther than the majority criterion: he identified a Condorcet winning candidate who placed last in a plurality vote and a Condorcet losing candidate who placed first. "So the candidate who really had plurality support is the very one who by the conventional method received the fewest votes. And [another candidate], who would have obtained the most votes by the conventional method, is in fact the candidate who is furthest from obtaining plurality support."[31] This is the heart of his argument for a new system, and Condorcet has here laid out another requirement for a reasonable electoral system, that any Condorcet winning candidate must win.[32] We today call this the "Condorcet criterion" and teach it as another logical condition which students must check for each voting system, just as Condorcet did. Our students are both very confident about this criterion when looking directly at a Condorcet tournament's graph and also very diffident about it when trying to formulate it correctly, as we will discuss below.

Condorcet adopted a mathematically modern point of view in considering two candidates in isolation and insisting their relative rankings there should match their relative rankings in the final outcome. Today, we denote this property "transitivity" or "independence," and it underlies much of the structure of algebra and other mathematical fields. The idea is implicit in the 1785 *Essai,* although Condorcet makes it clearer in 1788 in his criticism of the Borda count: "The points method confuses votes comparing Peter and Paul with those comparing either Peter or Paul to James and uses them to judge the relative merits of Peter and Paul. As long as it relies on irrelevant factors to form its judgments, it is bound to lead to error, and that is the real reason why this method is defective for a great many voting patterns."[33] We teach this criterion to our students, too, as a third principle; they tend to be more skeptical of it, though, due to some sort of innate acceptance that the relationships between people may be dependent on their social context and so not fully understood by examining them in isolation. On the one hand, they will agree that eliminating a loser should not change the outcome of an election; on the other, they will assert that it is OK if they cannot calculate the relative success of two candidates without considering the impact of all other candidates. They may also have become inured to violations of this criterion, though, due to the fact that the plurality system frequently violates it (as seen, in e.g., several recent U.S. Presidential popular votes).

Many more criteria for fairness were introduced by later scholars. Among these are the fourth criterion we teach, the "monotonicity criterion," which states that, if a candidate's standing improves with any one voter, then the candidate's overall standing in the election should not suffer.[34] Students do not internalize this criterion as well as the others, perhaps because it is so abstract: the most concrete examples we give in class involve, say, a last-minute political ad improving a candidate's standing with a few voters, which hurts that candidate's overall standing in an instant-runoff election. The students do express concern about this outcome, but they seem to feel that the complex interplay of all factors leading up to an election somehow overshadows the small effects of that ad changing those voters' preferences.

We teach these four criteria as the "fairness criteria." Tragically, Arrow's eponymous theorem tells us that these (along with several others) are not simultaneously attainable for any system beyond the majority system with two candidates.[35] Even when students disagree with the value placed on particular criteria, they do interact with them as interacting (and mutually incompatible) conditions that reasonable people may wish to put on their voting systems, and they engage with this example of how mathematics may inform and be informed by our world.

On logic and proofs

While exploring our fairness criteria, we quietly introduce logic and its natural expression, proof-writing. In order to check whether a system satisfies a criterion, students must first understand how a conditional statement works. "If P, then Q," has two possible scenarios: either the conclusion Q is true, or the hypothesis P is false (or possibly both); there is, in fact, only one situation which is forbidden, that P is true but Q is false. There are several logical fallacies students demonstrate commonly in this section. We attempt to trigger each one in class and then address it in the hopes that students will avoid them later in life.

For example, if we ask students to state the Condorcet criterion, a full-credit answer could be, "If there is a Condorcet winning candidate, then that candidate must actually be a winner."[36] Students sometimes confuse the statement with its converse, "The election winner must be a Condorcet winner."[37] They also may forget the case where the hypothesis is false, that is, where there is no such candidate: "The Condorcet winner will win." This is complicated by the fact that I, as a mathematician, would consider it acceptable to say, "A Condorcet winner will win," because I hear existence implied in *the* but not in *a*;[38] to my students, though, this is a questionable distinction, and many of them would initially consider these two statements equivalent. Similarly, a few students miss the distinction between a candidate being just one member of a set and being the whole set, i.e., being *one of* the winners vs. being *the* winner: "If there is a Condorcet winning candidate, then they beat everyone else." That said, as they must learn, they could legitimately rephrase the criterion as a condition on individual candidates rather than on the set of candidates, "If a candidate is a Condorcet winning candidate, then they must be an election winner," or they could even convert that statement to its contrapositive, "If a candidate loses, then that candidate must not be a Condorcet winning candidate."[39] Students also sometimes confuse the algorithm to calculate the outcome of a vote with the process of verifying whether the system satisfies a criterion. For example, we ask them what is wrong with stating, "The plurality system does not satisfy the Condorcet criterion because it just takes the candidate with the most votes, it doesn't compare each pair of candidates, so there can't be a Condorcet winner."

We unavoidably also spend some time discussing what constitutes a convincing mathematical argument. Students must proceed logically from a starting point to a conclusion, with each step following reasonably from the ones that came before. Simultaneously, they must learn to judge the background of their audience and determine how self-evident their statements are so that they employ an appropriate vocabulary and level of detail. For example, if we wish to demonstrate that Copeland's method satisfies the Condorcet criterion, we may argue that, if a candidate is a Condorcet winning candidate, then that candidate must have dominated every other candidate in the Condorcet tournament; in particular, no other candidate has dominated as often because each has been dominated at least once, by this candidate. Therefore, the Condorcet winning candidate will be the unique winner under Copeland's method. We could alternately establish the contrapositive: if some candidate loses under Copeland's method, then someone else has dominated more opponents in the tournament, and this candidate must not have dominated all of its opponents. Thus, it cannot be a Condorcet winning candidate. On the other hand, if we wish to show that the plurality method does not satisfy the Condorcet criterion, we have merely to reference Condorcet's first example above, where C is a Condorcet winning candidate yet lost the plurality vote to A.

These proofs also teach the difference between the roles played by a specific concrete example and a general logical argument. When we proved that Copeland's method satisfies the Condorcet criterion, both our arguments started with a hypothesis and proceeded logically to a conclusion. A single example where hypothesis and conclusion were both true would be insufficient (in fact, we could even find such an example for the plurality system: a Condorcet winning candidate often wins under a plurality vote). On the other hand, when showing that the plurality method does not satisfy the Condorcet criterion, a single counterexample sufficed, where the hypothesis was true, but not the conclusion. A logical argument was not required, nor would it always be advisable.

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Condorcet was the first to articulate criteria for an electoral system to fairly represent the views of its participants. He wrote extensively on collective decision-making not only as a mathematical descriptor subject to all the accompanying analysis and logical examination but also as an act which occurs in the life of a people and which must be responsive to their needs. We cover voting with our students because it is important to their lives and because we wish them to be knowledgeable and reflective citizens. At the same time, we quietly rejoice that this study accomplishes important educational objectives of logic and mathematical reasoning, as well as promoting historical literacy. Overall, learning about voting systems and fairness criteria helps our students follow the path Condorcet first trod, to formulate mathematical principles about how a voting system ought reasonably to translate individual preferences into a collective decision.

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[1] One of the milder marginal annotations in John Adams's copy of Condorcet's works is, "Thou art a Quack, Condorcet." Zoltán Haraszti, "John Adams Flays a Philosophe: Annotations on Condorcet's Progress of the Human Mind," *The William and Mary Quarterly* 7, no. 2 (1950): 235.

[2] We use a collection of units prepared by several professors and provided open source to the math community. David Lippman et al., *Math in Society* (ed. 2.5, 2017). About half of the textbooks and curriculum packets that arise in an online search include similar units on voting theory.

[3] Janet Heine Barnett, "The French Connection: Borda, Condorcet and the Mathematics of Voting Theory," *Convergence* 17 (September 2020), https://www.maa.org/press/periodicals/convergence/the-french-connection-borda-condorcet-and-the-mathematics-of-voting-theory. *Convergence* is the monthly publication of the Mathematical Association of America, the primary professional association for mathematics education in the U.S.

[4] Our source for technical material was Sherif El-Helaly, *The Mathematics of Voting and Apportionment: An Introduction* (Cham: Birkhäuser, 2019). We also read the historical texts mentioned in this article, along with a few more.

[5] Iain McLean, "The strange history of social choice, and the contribution of the Public Choice Society to its fifth revival," *Public Choice* 163 (2015): 153-165. Section 3 contains an executive summary of the history of voting theory.

[6] There are a few earlier references which show an understanding of how different voting systems may have different outcomes. For example, Pliny the Younger wrote about trying to convert two successive majority elections to a single plurality election, which would have changed the outcome. "Letter to Titius Aristo. A.D. 105," *Epistulae VIII.XIV*, in *Classics of Social Choice,* edited and translated by Iain McLean and Arnold Urken (Ann Arbor: University of Michigan Press, 1995), 71-73.

[7] In the Catalan novel *Blanquerna* (c. 1283), excerpt in *Classics of Social Choice,* McLean and Urken, 71-73; in two Latin pamphlets, "Artifitium electionis personarum" ("The art of the election of persons," before 1283), in "Llull's writings on electoral systems," edited and translated by Günter Hägele and Friedrich Pukelsheim, *Studia Lulliana 41* (2001): 3–38, and "De arte electionis" ("On the art of elections," 1299), excerpt in *Classics of Social Choice,* McLean and Urken, 71-73.

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[8] *De concordantia Catholica* (*On Catholic harmony*, 1434), excerpt in *Classics of Social Choice,* McLean and Urken, 77-78.

[9] The two pamphlets foreshadow Condorcet's pairwise comparison method; the *Blanquerna* passage is difficult to interpret, but it seems to be Borda's. Cusanus appears to have been unaware of *Blanquerna* and also discovered Borda's method independently. For "De arte electionis" and *De condordantia Catholica,* see McLean and Urken, *Classics of Social Choice,* 16-22; for "Artifitium," see Hägele and Pukelsheim, "Llull's writings," 7-11; for *Blanquerna,* see Josep Colomer, "Ramon Llull: From 'ars electionis' to social choice theory." *Social Choice and Welfare* 40, no. 2 (2013): 317-328.

[10] "Preliminary Discussion, Analysis of Part One, Fourth Example: An Election Between Three Candidates," *Essai* lvi-lxx. Condorcet also discusses electoral form briefly elsewhere; see excerpts, in translation, in *Condorcet: Foundations of Social Choice and Political Theory,* edited and translated by Iain McLean and Fiona Hewitt (Cheltenham: Edward Elgar, 1994), 111-113, 131-252.

[11] See "On Fairness," below.

[12] "Mémoire sur les élections au scrutin" ("A paper on elections by ballot"), *Histoire de l'Académie Royale des Sciences, Anée 1781* Paris, 1784, in *Classics of Social Choice,* McLean and Urken, 83-89. The Academy's *Procés-verbaux* indicate that Condorcet announced his book at a meeting on 14 July 1784 and Borda read his "Mémoire" 21 July 1784. The preface to the printed edition of Borda's "Mémoire" claims his ideas had been presented to the Academy in 1770, although the *Procés-verbaux* do not mention a formal reading at that time. Condorcet, presumably to provide Borda priority without delaying his own manuscript, arranged for Borda's paper to be included in the 1781 edition of the Academy's proceedings, which was then under preparation in 1784, and his own *Essai* appeared in 1785. For historiography, see Eric Brian, "Condorcet and Borda in 1784. Misfits and Documents," *Journal Électronique d'Histoire des Probabilités et de la Statistique,* 4, no. 1 (June 2008), https://www.jehps.net/juin2008/Brian.pdf.

[13] According to McLean, the copy of Borda's essay in the library at Christ Church was uncut; that library did not contain Condorcet's work, but the main Bodleian Library did, and that book, also, was uncut when seen by the historian Black in the 1940s, although someone has since cut it. See Iain McLean, "Voting," in *The Mathematical World of Charles L. Dodgson (Lewis Carroll)*, ed. Robin Wilson and Amirouche Moktefi (Oxford: Oxford University Press, 2019), 135.

[14] Charles Dodgson, "A discussion of the various methods of procedure in conducting elections" (1873), "Suggestions as to the best method of taking votes, where more than two issues are to be voted on." (1874), "A method of taking votes on more than two issues," (1876); all in *The Classics of Social Choice*, McLean and Urken, 279-297.

[15] Condorcet, *Essai sur la constitution,* in *Condorcet,* McLean and Hewitt, 149; Thomas Hare, *A Treatise on the Election of Representatives, Parliamentary and Municipal* (London: Longman, Green, Longman, and Roberts, 1861); E. J. Nanson, "Methods of election: Ware's Method," *Transactions and Proceedings of the Royal Society of Victoria* 19 (1882): 197-240. Ware seems not to have written about it; Hare and Nanson attribute the idea to him.

[16] As he admitted in the notes to a later edition of his ground-breaking paper, he had perhaps not completed the most thorough search of the previous literature. Kenneth Arrow, "Social choice and individual values," (New Haven: Yale University Press, Cowles Foundation Monograph no. 12, 2nd ed, 1963).

[17] For an overview of the history and current questions in social choice theory, see Christian List, "Social Choice Theory," *The Stanford Encyclopedia of Philosophy* (Spring 2022 Edition), ed. Edward N. Zalta, https://plato.stanford.edu/archives/spr2022/entries/social-choice/.

[18] "Social choice and individual values," 13 (in the 1951 edition).

[19] *Essai,* in *Condorcet,* McLean and Hewitt, 123.

[20] Llull, *Blanquerna* (this is a very abstruse presentation - Cusanus appears to have been unaware of it); Cusanus, *De concordantia Catholica;* Borda, "Mémoire"; Dodgson, 1873 and 1876 pamphlets.

[21] "Mémoire," *Classics of Social Choice,* McLean and Urken, 84. The "conventional election method" is the plurality system; "obtain the plurality" refers to obtaining some kind of collective endorsement, not winning a plurality vote.

[22] Condorcet, *Essai sur la constitution*, in *Condorcet*, 149; Hare, "Treatise"; Nanson, "Methods of election."

[23] Llull, "Artifitium" and "De arte electionis"; Condorcet, *Essai,* in *Condorcet,* McLean and Hewitt, 123; Dodgson, 1873 and 1876 pamphlets; Arthur Copeland, "A 'reasonable' social welfare function," Seminar on mathematics in the social sciences, University of Michigan, 1951, known from R. Duncan Luce and Howard Raiffa, *Games and Decisions: Introduction and Critical Survey* (New York: Wiley, 1957), 358.

[24] *Essai,* in *Condorcet,* McLean and Hewitt, 124.

[25] E.g., if there were two more candidates D and E who respectively beat B and A but lost to C, then C would dominate the most pairwise comparisons and be the winner; there is a preference schedule where this is attainable.
[26] Condorcet *Essai,* in *Condorcet,* McLean and Hewitt, 129.
[27] "Mémoire," in *Classics of Social Choice,* McLean and Urken, 84.
[28] *Essai,* in *Condorcet,* McLean and Hewitt, 97.
[29] For technical reasons found in, e.g., the proof of Arrow's Theorem, the modern version of the majority criterion requires that any majority winning candidate be among the winners, not necessarily the sole winner.
[30] Borda's and Condorcet's criticism of the plurality system is sometimes called the "majority loser criterion," and both their systems do satisfy it, as does instant runoff.
[31] *Essai*, in *Condorcet,* McLean and Hewitt, 123. As in Borda's writing, the "conventional method" is the plurality method, and "plurality support" refers to the collective will of the group, not the outcome of the plurality method.
[32] As with the majority criterion, the modern formulation of the Condorcet criterion requires any Condorcet winning candidate be among the winners, not be the only winner.
[33] *Essai sur la constitution,* in *Condorcet,* McLean and Hewitt, 152.
[34] See El-Helaly, *Mathematics of Voting;* it covers monotonicity in 1.2.4 along with a few of the other commonly discussed criteria like weak Pareto efficiency in 1.3.6.
[35] "Social choice," Theorem 2, p. 59.
[36] This statement could properly be criticized as too casual outside the classroom environment; a more formal statement might be something like, "If there is a Condorcet winning candidate, then that candidate must be among the winners," or perhaps, "Criterion C: If A is a Condorcet winning candidate, then A is a member of the winning set."
[37] An astute reader will note that this is not a true converse; a converse of Criterion C above would be: "If A is a member of the winning set, then A is a Condorcet winning candidate."
[38] In slightly more mathematical language, this is the difference between "There exists an A such that A is a Condorcet winning candidate and A is a winner," and "For any A such that A is a Condorcet winning candidate, then A is a winner."
[39] Again, the astute reader will likely render the contrapositive of Criterion C as, "If A is not a member of the winning set, then A is not a Condorcet winning candidate."

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