Essential Extensions and Injective Hulls of Fuzzy Modules

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Abstract

In this paper, we introduce the notion of essential extensions of fuzzy modules. We use these concepts to introduce the notion of injective hulls of fuzzy modules. It is known that every R-module has an injective hull, where R is a ring. We show that these corresponding results do not hold for fuzzy R-modules, i.e., there exists a fuzzy R-module that does not have an injective hull. Sufficient conditions are given for a fuzzy R-module to have an injective hull.

Keywords: Algebra, fuzzy module, fuzzy homomorphism, essential extension, injective hull.

1 Introduction

The theory of fuzzy sets was initiated by Zadeh [6] in 1965. Rosenfeld [4] introduced the concept of fuzzy subgroups of a group in 1971. Since then various algebraic structures have been fuzzified. Pan [3], Lopez and Malik [2] and Zahedi and Ameri [7,8] studied fuzzy modules of a ring. Injective and projective fuzzy modules were characterized by Lopez and Malik in [2]. These modules were later studied in [7,8]. In this paper, we first introduce the notion of essential extensions of fuzzy modules in Section 3. We obtain basic properties of essential extensions of fuzzy modules. We apply these concepts to introduce the notion of injective hulls of fuzzy modules over a ring in Section 4. It is known that every module over a ring has an injective hull [1, 5]. We show that these analogous concepts fail to hold for fuzzy modules, i.e., there exists a fuzzy module for which an injective hull does not exist. We give sufficient conditions for a fuzzy module to have an injective hull.

2 Preliminary and Notations

Let A be a nonempty set. A **fuzzy subset** μ of A is a function $\mu : A \to [0, 1]$.

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Let μ be a fuzzy subset of A. Let $t \in [0,1]$. Then $\mu_t = \{x \in A \mid \mu(x) \geq t\}$ is called a t-**cut** or a t-**level subset** of A. The set $\mu^* = \{x \in A \mid \mu(x) > 0\}$ is called the **support** of μ.

Let a_i, $i \in \Lambda$ be real numbers, where Λ is finite or infinite. Then

$$\begin{aligned} \wedge_{i\in\Lambda} a_i &= \inf\{a_i \mid i \in \Lambda\}, \text{ and} \\ \vee_{i\in\Lambda} a_i &= \sup\{a_i \mid i \in \Lambda\}. \end{aligned}$$

If Λ is finite, then typically, $\inf\{a_i \mid i \in \Lambda\}$ is written as $\min\{a_i \mid i \in \Lambda\}$ and $\sup\{a_i \mid i \in \Lambda\}$ is written as $\max\{a_i \mid i \in \Lambda\}$.

Let R be a ring. A **fuzzy left** (**right**) R-module is defined to be a pair (M, μ), where M is a left (right) R-module and $\mu : M \to [0,1]$ is a function satisfying

(i) $\mu(x + y) \geq \mu(x) \wedge \mu(y)$,

(ii) $\mu(-x) = \mu(x)$,

(iii) $\mu(0) = 1$, and

(iv) $\mu(rx) \geq \mu(x) \qquad (\mu(xr) \geq \mu(x))$

for all $x, y \in M$ and $r \in R$.

Unless otherwise stated, all (fuzzy) R-modules in this paper are (fuzzy) left R-modules. For any R-module M, let

$$\theta : M \to [0,1]$$

be defined by

$$\theta(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x \neq 0 \end{cases}$$

for all $x \in M$. Then (M, θ) is fuzzy R-module.

Define

$$\mathbf{1}_M : M \to [0,1]$$

by

$$\mathbf{1}_M(x) = 1$$

for all $x \in M$. Then $(M, \mathbf{1}_M)$ is a fuzzy R-module.

Let α and β be fuzzy subsets of a set A. We define $\alpha \leq \beta$ if $\alpha(x) \leq \beta(x)$ for all $x \in A$. If B is a subset of A, then χ_B denotes the characteristic function of B.

Let (N, α) and (M, μ) be fuzzy R-modules. (N, α) is **fuzzy R-submodule** of (M, μ), written $(N, \alpha) \subseteq (M, \mu)$, if N is an R-submodule of M and $\alpha \leq \mu|_N$. (N, α) is called **nonzero** if $N \neq \{0\}$ and $\alpha \neq \theta$. It is easy to check that (N, α) is nonzero if and only if $\alpha \neq \theta$.

For $\mu : M \to [0,1]$, let $\mathrm{Im}(\mu)$ denote the image of μ.

Suppose $(N, \alpha) \subseteq (M, \mu)$. Define $\alpha' : M \to [0,1]$ by $\alpha'(x) = \alpha(x)$ if $x \in N$ and $\alpha'(x) = 0$ otherwise. Then (M, α') is a fuzzy R-submodule of (M, μ) and $(N, \alpha) \subseteq (M, \alpha') \subseteq (M, \mu)$. Now $\alpha \leq \mu|_N$ if and only if $\alpha' \leq \mu$. Thus if no confusion arises, then whenever we write $\alpha \leq \mu$ we mean $\alpha \leq \mu|_N$ or $\alpha' \leq \mu$.

Let $\{N_i \mid i \in \Lambda\}$ be a family of subsets of M and $\alpha_i : N_i \to [0,1]$ for all $i \in \Lambda$. Define $\cap_{i\in\Lambda}\alpha_i : \cap_{i\in\Lambda}N_i \to [0,1]$ by $(\cap_{i\in\Lambda}\alpha_i)(x) = \wedge_{i\in\Lambda}\alpha_i(x)$ for all $x \in \cap_{i\in\Lambda}N_i$. If $\{(N_i, \alpha_i) \mid i \in \Lambda\}$ is a family of fuzzy R-submodules of (M, μ), then $(\cap_{i\in\Lambda}N_i, \cap_{i\in\Lambda}\alpha_i)$ is a fuzzy R-submodule of (M, μ).

Let N and K be subsets of M and $\alpha : N \to [0,1]$ and $\gamma : K \to [0,1]$. Define $\alpha + \gamma : N + K \to [0,1]$ by

$$(\alpha + \gamma)(x) = \vee\{\alpha(u) \wedge \gamma(v) \mid x = u + v \text{ for some } u \in N \text{ and } v \in K\}.$$

If (N, α) and (K, γ) are fuzzy R-submodules of (M, μ), then $(N + K, \alpha + \gamma)$ is a fuzzy R-submodule of (M, μ).

Proposition 1 *Let M be an R-module and $\mu : M \to [0,1]$. Then (M, μ) is a fuzzy R-module if and only if μ_t is an R-submodule of M for all $t \in [0,1]$.* ■

Proposition 2 *Let M be an R-module and $\mu : M \to [0,1]$. Then μ^* is an R-module.* ■

Proposition 3 *Let (M, μ) be a fuzzy R-module. Let $x, y \in M$ and $\mu(x) > \mu(y)$. Then $\mu(x - y) = \mu(y)$.* ■

Let A and B be sets and $f : A \to B$ be a mapping. Let α be a fuzzy subset of A and β be a fuzzy subset of B. Define the fuzzy subsets $f(\alpha)$ of B and $f^{-1}(\beta)$ of A by

$$f(\alpha)(b) = \begin{cases} \vee\{\alpha(a) \mid a \in A \text{ and } f(a) = b\}, & \text{if } f^{-1}(b) \neq \emptyset \\ 0 & \text{if } f^{-1}(b) = \emptyset, \end{cases}$$

and

$$f^{-1}(\beta)(a) = \beta(f(a))$$

for all $a \in A$ and $b \in B$.

Definition 4 *Let (M, μ) and (N, η) be fuzzy R-modules and $f : M \to N$ be an R-homomorphism. Then f is called a* ***fuzzy R-homomorphism*** *from (M, μ) into (N, η) if $\eta(f(x)) \geq \mu(x)$ for all $x \in M$. We express this fact symbolically with the notation $\bar{f} : (M, \mu) \to (N, \eta)$.*

Given a fuzzy R-homomorphism $\bar{f} : (M, \mu) \to (N, \eta)$, $f : M \to N$ is called the **underlying R-homomorphism** of $\bar{f}$. Similarly M is the underlying module (set) of (M, μ). $\bar{f}$ is called a **monomorphism** if f is a monomorphism.

Given a ring R (not necessarily with 1), the classes of all fuzzy R-modules and fuzzy R-homomorphism constitutes the objects and morphisms respectively of a category R-fzmod, where composition of morphisms is the usual composition of functions [2].

3 Essential Extensions

In this section, we define an essential extension of a fuzzy module and obtain its basic properties. We will use these concepts in the next section.

Definition 5 *Let (N, η) be a fuzzy R-submodule of (M, μ). (N, η) is called* ***essential*** *in (M, μ) (or (M, μ) is called an* ***essential extension*** *of (N, η)), written $(N, \eta) \subseteq_e (M, \mu)$, if for every nonzero fuzzy R-submodule (A, α) of (M, μ), $N \cap A \neq \{0\}$ and $\eta \cap \alpha \neq \theta$.*

Let (M, μ) be a fuzzy R-module. Clearly $(M, \mu) \subseteq_e (M, \mu)$ and if $(N, \eta) \subseteq_e (M, \mu)$ and (M, μ) is nonzero, then $N \neq \{0\}$ and $\eta \neq \theta$. When we write $\eta \subseteq_e \mu$, we mean that $\eta \cap \alpha \neq \theta$ for every nonzero fuzzy subset α of M such that $\alpha \leq \mu$.

Definition 6 *Let $\bar{f} : (M, \mu) \to (N, \eta)$ be a fuzzy R-monomorphism. $\bar{f}$ is called an* ***essential monomorphism*** *if $(f(M), f(\mu)) \subseteq_e (N, \eta)$.*

Proposition 7 *Let (N, η) be a fuzzy R-submodule of (M, μ). Then $(N, \eta) \subseteq_e (M, \mu)$ if and only if $N \subseteq_e M$ and $\eta^* \subseteq_e \mu^*$.*

Proof. We only need to verify that $\eta \subseteq_e \mu$ if and only if $\eta^* \subseteq_e \mu^*$. Suppose $\eta \subseteq_e \mu$. Let A be a nonzero submodule of μ^*. Define $\gamma(x) = \mu(x)$ if $x \in A$ and $\gamma(x) = 0$, otherwise. Then (A, γ) is a nonzero fuzzy R-submodule of (M, μ). Hence $\gamma \cap \eta \neq \theta$ and so there exists $x \neq 0$ such that $(\gamma \cap \eta)(x) \neq 0$. Thus $x \in \gamma^* \cap \eta^* = A \cap \eta^*$. Hence $\eta^* \subseteq_e \mu^*$. Conversely, suppose $\eta^* \subseteq_e \mu^*$. Let $\gamma \neq \theta$ and $\gamma \subseteq \mu$. Then $\gamma^* \neq \{0\}$. Thus γ^* is a nonzero R-submodule of μ^*. Hence $\gamma^* \cap \eta^* \neq \{0\}$. Let $0 \neq x \in \gamma^* \cap \eta^*$. Then $(\gamma \cap \eta)(x) = \gamma(x) \wedge \eta(x) \neq 0$. Hence $\gamma \cap \eta \neq \theta$. Consequently, $\eta \subseteq_e \mu$. ■

Proposition 8 *Let (N, η) be a fuzzy R-submodule of (M, μ). If $N \subseteq_e M$ and $\eta_t \subseteq_e \mu_t$ for all $t \in [0, 1]$, then $(N, \eta) \subseteq_e (M, \mu)$.*

Proof. Let $\gamma \neq \theta$ and $\gamma \leq \mu$. There exists $x \neq 0$ such that $\gamma(x) = t \neq 0$. Thus γ_t is a nonzero R-submodule of μ_t. Since $\eta_t \subseteq_e \mu_t$, $\eta_t \cap \gamma_t \neq \{0\}$. Let $a \neq 0$ and $a \in \eta_t \cap \gamma_t$. Then $(\eta \cap \gamma)(a) = \eta(a) \wedge \gamma(a) \geq t \neq 0$. Hence $\eta \cap \gamma \neq \theta$. Thus $(N, \eta) \subseteq_e (M, \mu)$. ■

The following example shows that the converse of Proposition 8 is not true.

Example 9 *$\mathbb{Z}_6$ is a $\mathbb{Z}$-module. Let $A = \{\bar{0}, \bar{2}, \bar{4}\}$ and $B = \{\bar{0}, \bar{3}\}$. Then A and B are $\mathbb{Z}$-submodules of $\mathbb{Z}_6$. Define $\delta : \mathbb{Z}_6 \to [0, 1]$ by*

$$\delta(x) = \begin{cases} 1 & \text{if } x = \bar{0} \\ \frac{1}{2} & \text{if } x \in A \backslash \{\bar{0}\} \\ \frac{1}{3} & \text{if } x \in \mathbb{Z}_6 \backslash A. \end{cases}$$

Then $(\mathbb{Z}_6, \delta)$ is a fuzzy $\mathbb{Z}$-submodule of $(\mathbb{Z}_6, \mathbf{1}_{\mathbb{Z}_6})$. Let (K, γ) be a nonzero submodule of $(\mathbb{Z}_6, \mathbf{1}_{\mathbb{Z}_6})$. Then $\mathbb{Z}_6 \cap K = K \neq \{0\}$. There exists $a \in K$ such that $\gamma(a) \neq 0$. Now $\delta(a) \geq \frac{1}{3}$. Hence $(\delta \cap \gamma)(a) = \delta(a) \wedge \gamma(a) \neq 0$. Thus $\delta \cap \gamma \neq \theta$. Hence $(\mathbb{Z}_6, \delta) \subseteq_e (\mathbb{Z}_6, \mathbf{1}_{\mathbb{Z}_6})$. Since $\delta_{\frac{1}{2}} \cap B = \{\bar{0}\}$, it follows that $\delta_{\frac{1}{2}}$ is not essential in $\mathbf{1}_{\frac{1}{2}} = \mathbb{Z}_6$.

Theorem 10 *Let (K,γ), (N,η) and (M,μ) be fuzzy R-modules such that $(K,\gamma) \subseteq (N,\eta) \subseteq (M,\mu)$.*

(i) $(K,\gamma) \subseteq_e (M,\mu)$ if and only if $(K,\gamma) \subseteq_e (N,\eta)$ and $(N,\eta) \subseteq_e (M,\mu)$.

(ii) Suppose $(K,\gamma) \subseteq_e (N,\eta) \subseteq (M,\mu)$ and $(K',\gamma') \subseteq_e (N',\eta') \subseteq (M,\mu)$. Then $(K \cap K', \gamma \cap \gamma') \subseteq_e (N \cap N', \eta \cap \eta')$.

(iii) Let $\bar{f} : (M,\mu) \to (M',\mu')$ be a fuzzy R-monomorphism. Let (A,α) and (B,β) be fuzzy R-submodules of (M',μ'). If $(A,\alpha) \subseteq_e (B,\beta)$, then $(f^{-1}(A), f^{-1}(\alpha)) \subseteq_e (f^{-1}(B), f^{-1}(\beta))$, where $f^{-1}(\alpha)(x) = \alpha(f(x))$ for all $x \in M$,

Proof. (i) Suppose $(K,\gamma) \subseteq_e (M,\mu)$. Let (A,δ) be a nonzero fuzzy R-submodule of (N,η). Then (A,δ) is a nonzero fuzzy R-submodule of (M,μ). Hence $A \cap K \neq \{0\}$ and $\gamma \cap \delta \neq \theta$. Thus $(K,\gamma) \subseteq_e (N,\eta)$. Also, if (B,β) is a nonzero fuzzy R-submodule of (M,μ), then $B \cap N \supseteq B \cap K \neq \{0\}$ and $\beta \cap \eta \supseteq \beta \cap \gamma \neq \theta$. Hence $(N,\eta) \subseteq_e (M,\mu)$. Conversely, suppose that $(K,\gamma) \subseteq_e (N,\eta)$ and $(N,\eta) \subseteq_e (M,\mu)$. Let (B,β) is a nonzero fuzzy R-submodule of (M,μ). Then $B \cap N \neq \{0\}$ and $\beta \cap \eta \neq \theta$. Thus $(B \cap N, \beta \cap \eta)$ is a nonzero fuzzy R-submodule of (N,η). Hence $K \cap B = K \cap B \cap N \neq \{0\}$ and $\beta \cap \gamma = \gamma \cap \beta \cap \eta \neq \theta$. Hence $(K,\gamma) \subseteq_e (M,\mu)$.

(ii) Let (A,α) be a nonzero fuzzy R-submodule of $(N \cap N', \eta \cap \eta')$. Then (A,α) is a nonzero fuzzy R-submodule of (N,η). Thus $A \cap K \neq \{0\}$ and $\gamma \cap \alpha \neq \theta$. Thus $(A \cap K, \gamma \cap \alpha)$ is a nonzero and clearly also a fuzzy R-submodule of (N',η'). Hence $A \cap K \cap K' \neq \{0\}$ and $\gamma \cap \alpha \cap \gamma' \neq \theta$. Thus $(K \cap K', \gamma \cap \gamma') \subseteq_e (N \cap N', \eta \cap \eta')$.

(iii) Let (K,γ) be a nonzero fuzzy R-submodule of $(f^{-1}(B), f^{-1}(\beta))$. Since f is one-one, $f(K)$ is a nonzero submodule of B. Now $f(\gamma)(y) = \vee\{\gamma(x) \mid f(x) = y,\, x \in K\} \leq \vee\{f^{-1}(\beta)(x) \mid f(x) = y,\, x \in K\} = \vee\{\beta(f(x)) \mid f(x) = y,\, x \in K\} = \beta(y)$. Hence $f(\gamma) \leq \beta$. Suppose $f(\gamma) = \theta$. Let $x \in K$ and $x \neq 0$. Since f is one-one $f(x) \neq 0$. Now $f(\gamma)(f(x)) = \vee\{\gamma(u) \mid f(u) = f(x),\, u \in K\} = \gamma(x)$ (since f is one-one) and so $\gamma(x) = 0$. This implies that $\gamma = \theta$, a contradiction. Hence $f(\gamma) \neq \theta$. Thus $(f(K), f(\gamma))$ is a nonzero R-submodule of (B,β). Hence $f(K) \cap A \neq \{0\}$ and $f(\gamma) \cap \alpha \neq \theta$. Let $y \neq 0$ and $y \in f(K) \cap A$. Then $y = f(x)$ for some $x \in K$. Since f is one-one and $y \neq 0$, $x \neq 0$. Now $x \in K \cap f^{-1}(A)$. Since $f(\gamma) \cap \alpha \neq \theta$, there exists $0 \neq y \in A$ such that $(f(\gamma) \cap \alpha)(y) \neq 0$. Thus $f(\gamma)(y) \neq 0$ and $\alpha(y) \neq 0$. Thus $\vee\{\gamma(x) \mid f(x) = y,\, x \in K\} \neq 0$ Thus $\gamma(x) \neq 0$ for some $x \in K$, $f(x) = y$. Now $\alpha(y) \neq 0$ implies that $\alpha(f(x)) \neq 0$ and so $(f^{-1}(\alpha))(x) \neq 0$. Since $y \neq 0$ and f is one-one, it follows that $x \neq 0$. Hence $\gamma \cap f^{-1}(\alpha) \neq \theta$. Thus $(f^{-1}(A), f^{-1}(\alpha)) \subseteq_e (f^{-1}(B), f^{-1}(\beta))$. ■

In the following example we show that an infinite intersection of essential fuzzy R-submodules need not be essential.

Example 11 *For all $n \in \mathbb{N}$, define $\alpha_n : \mathbb{Z} \to [0,1]$ by*

$$\alpha_n(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{n} & \text{if } x \in n\mathbb{Z}\backslash\{0\} \\ 0 & \text{otherwise.} \end{cases}$$

Then for all $n \in \mathbb{N}$, $(\mathbb{Z}, \alpha_n)$ is a fuzzy R-submodule of $(\mathbb{Z}, 1_{\mathbb{Z}})$. Let $\alpha = \cap_{n\in\mathbb{N}}\alpha_n$. Then $\cap_{n\in\mathbb{N}}(\mathbb{Z}, \alpha_n) = (\mathbb{Z}, \alpha)$ is a fuzzy R-submodule of $(\mathbb{Z}, 1_{\mathbb{Z}})$. Let (K,γ) be a *nonzero fuzzy R-submodule of* $(\mathbb{Z}, 1_{\mathbb{Z}})$. *Clearly,* $K \cap \mathbb{Z} \neq \{0\}$. *Let* $0 \neq m \in \mathbb{Z}$ *be such that* $\gamma(m) \neq 0$. *Then* $\gamma(mn) \geq \gamma(m) \neq 0$. *Also* $\alpha_n(mn) \neq 0$. *Hence* $\alpha_n \cap \gamma \neq \theta$. *Thus for all* $n \in \mathbb{N}$, $(\mathbb{Z}, \alpha_n) \subseteq_e (\mathbb{Z}, 1_{\mathbb{Z}})$. *Let* $x \in \mathbb{Z}$ *and* $x \neq 0$. *Let* $\alpha(x) = t$. *This implies that* $\frac{1}{n} \geq t$ *for all* n *and so* $t = 0$. *Thus* $\alpha = \theta$. *Consequently,* $\cap_{n\in\mathbb{N}}(\mathbb{Z}, \alpha_n)$ *is not essential in* $(\mathbb{Z}, 1_{\mathbb{Z}})$.

Example 12 *Let* $R = \mathbb{Z}$ *and* $M = \mathbb{Z} \oplus \mathbb{Z}_2$. *Then* M *is an R-module. Let* $K = K' = (2, \overline{0})R$, $N = (1, \overline{0})R$ *and* $N' = (1, \overline{1})R$. *Then* K, K', N *and* N' *are R-submodules of* M. *Define* $\gamma : K \to [0, 1]$ *by*

$$\gamma(x) = \begin{cases} 1 & \text{if } x \in K \\ 0 & \text{otherwise.} \end{cases}$$

Let $\gamma' = \gamma$, $\eta = \chi_N$ *and* $\eta' = \chi_{N'}$. *Then* $(K, \gamma) \subseteq (N, \eta)$, $(K', \gamma') \subseteq (N', \eta')$ *are fuzzy R-submodules of* (M, χ_M).

Let (A, α) *be a nonzero fuzzy R-submodule of* (N, η). *Let* $x = n(1, \overline{0}) = (n, \overline{0})$ *be a nonzero element of* A. *Then* $2x \neq 0$ *and* $2x = (2n, \overline{0}) \in K \cap A$ *and so* $K \cap A \neq \{0\}$. *Since* $\alpha \neq \theta$, *there exists* $0 \neq x \in A$ *such that* $\alpha(x) \neq 0$. *Let* $x = t(1, \overline{0}) = (t, \overline{0})$ *for some* $t \in R$. *Now* $\alpha(2(t, \overline{0})) \geq \alpha((t, \overline{0})) \neq 0$ *and* $2(t, \overline{0}) \in K \cap A$. *Also* $\gamma(2(t, \overline{0})) = 1 \neq 0$. *Hence* $\alpha \cap \gamma \neq \theta$. *Thus* $(K, \gamma) \subseteq_e (N, \eta)$.

Let (B, β) *be a nonzero fuzzy R-submodule of* (N', η'). *Let* $x = n(1, \overline{1}) = (n, \bar{n})$ *be a nonzero element of* B. *Now* $2x = 2(n, \bar{n}) = (2n, \overline{0}) \in B \cap K'$. *Thus* $B \cap K' \neq \{0\}$. *Let* $0 \neq u \in B$ *be such that* $\beta(u) \neq 0$. *Now* $u = t(1, \overline{1})$ *for some integer* t. *Thus* $\beta(2u) \geq \beta(u) \neq 0$. *Also* $\gamma'(2u) = \gamma'(2t, \overline{0}) = 1$. *Hence* $\beta \cap \gamma' \neq \theta$. *Thus* $(K', y') \subseteq_e (N', \eta')$.

Let $T = (0, \overline{1})R$. *Define* $\delta : T \to [0, 1]$ *by*

$$\gamma(x) = \begin{cases} 1 & \text{if } x \in T \\ 0 & \text{otherwise.} \end{cases}$$

Then (T, δ) *is a nonzero fuzzy R-submodule of* (N', η') *and also of* (M, χ_M). *Thus* (T, δ) *is a nonzero fuzzy R-submodule of* $(N + N', \eta + \eta')$. *Now* $K + K' = K$ *and* $\gamma + \gamma' = \gamma$. *Since* $K \cap T = \{0\}$, *it follows that* $(K + K', \gamma + \gamma') \nsubseteq_e (N + N', \eta + \eta')$.

Proposition 13 *Let* (N, η) *be a nonzero fuzzy R-submodule of* (M, μ). *If* $\mu|_N \neq \eta$, *then there exists a fuzzy R-submodule* (A, α) *of* (M, μ) *such that* $(N, \eta) \subseteq_e (A, \alpha)$ *and* $(N, \eta) \neq (A, \alpha)$.

Proof. Let $A = N$. There exists $x \in N$, $x \neq 0$ such that $\mu(x) > \eta(x)$. Define

$$\alpha(x) = \begin{cases} \mu(x) & \text{if } x \in \eta^* \\ 0 & \text{otherwise.} \end{cases}$$

Then (A, α) is a fuzzy R-submodule of (M, μ). Clearly $(N, \eta) \neq (A, \alpha)$. Let (K, γ) be a nonzero fuzzy R-submodule of (A, α). Now $K \cap N = K \neq \{0\}$. There exists $x \neq 0$, $x \in K$ such that $\gamma(x) \neq 0$. Then $\alpha(x) \neq 0$ and so $x \in \alpha^* = \eta^*$. Thus $\eta(x) \neq 0$ and so $(\eta \cap \gamma)(x) \neq 0$. Hence $\eta \cap \gamma \neq \theta$. Thus $(N, \eta) \subseteq_e (A, \alpha)$. ■

Proposition 14 *Let (A, α) be a fuzzy R-module such that $A = \alpha^*$. Let (E, μ) be a fuzzy R-module such that $(A, \alpha) \subseteq (E, \mu)$. Then E is an essential extension of A if and only if then (E, μ) is an essential extension of (A, α).*

Proof. Suppose E is an essential extension of A. Let (K, γ) be a nonzero fuzzy R-submodule of (E, μ). Then K is a nonzero R-submodule of E and so $A \cap K \neq \{0\}$. Now since $\gamma \neq \theta$, γ^* is a nonzero R-submodule of K and hence of E. This implies that $A \cap \gamma^* \neq \{0\}$. Let $0 \neq a \in A \cap \gamma^*$. Then $\alpha(a) \neq 0$ and $\gamma(a) \neq 0$. Hence $(\alpha \cap \gamma)(a) \neq 0$. Thus (E, μ) is an essential extension of (A, α). The converse is trivial. ■

4 Injective Hulls of Fuzzy R-modules

In this section, we define and study the injective hulls of fuzzy R-modules. First we recall the following two results given in [5].

Proposition 15 (5) *Let M be an R-module and let E be an R-module such that E is an extension of M. The following are equivalent.*
(i) E is an essential extension of M.
(ii) E is a maximal essential extension of M.
(iii) E is a minimal injective extension of M.

Theorem 16 (5) *Let M be an R-module. Then there exists an R-module E satisfying the equivalent conditions of Proposition 15.*

Definition 17 (2) *A fuzzy R-module (E, μ) is called* ***injective*** *if (E, μ) is an injective object in R-fzmod.*

Theorem 18 (2) *Let (E, μ) be a fuzzy R-module. Then (E, μ) is injective if and only if E is an injective R-module and $\mu = \mathbf{1}_E$.* ■

Proposition 19 *Let (A, α) and (E, μ) are fuzzy R-modules and (E, μ) is injective. Let $f : (A, \alpha) \to (E, \mu)$ be a fuzzy R-monomorphism. Let (B, β) be a fuzzy R-module such that $(A, \alpha) \subseteq_e (B, \beta)$. Then there exists a fuzzy monomorphism $f' : (B, \beta) \to (E, \mu)$ such that f' is an extension of f.*

Proof. Let $i : (A, \alpha) \to (B, \beta)$ be the identity map. Since (E, μ) is injective, E is injective and $\mu = \mathbf{1}_E$. Consider the following diagram:

$$\begin{array}{lcl} (A, \alpha) & \xrightarrow{f} & (E, \mu) \\ i \downarrow & \nearrow f' & \\ (B, \beta) & & \end{array}$$

Since (E, μ) is injective, there exists a fuzzy R-homomorphism $f' : (B, \beta) \to (E, \mu)$ such that $f' \circ i = f$. Clearly f' is an extension of f. Since $(A, \alpha) \subseteq_e (B, \beta)$, $A \subseteq_e B$. Hence $A \cap$Ker f' =Ker $f = \{0\}$ since f is one-one. This implies that Ker $f' = \{0\}$. Hence f' is one-one and so f' is a fuzzy monomorphism. ■

Proposition 20 *Let (E, μ) be an injective fuzzy R-module and (K, γ) be a fuzzy R-submodule of (E, μ). If (K, γ) is a direct summand of (E, μ), then (K, γ) is injective.*

Proof. Let (K', γ') be a fuzzy R-submodule of (E, μ) such that $(E, \mu) = (K, \gamma) \oplus (K', \gamma')$. Then $E = K \oplus K'$ and $\mu = \gamma \oplus \gamma'$. Since E is injective, K is injective. Also $\mu = \mathbf{1}_E$. Let $x \in K$. Then $1 = \mu(x) = \vee\{\gamma(u) \wedge \gamma'(v) \mid x = u + v\} = \gamma(x) \wedge \gamma'(0)$ (since x has a unique representation as $x = x + 0$) $= \gamma(x)$. Hence $\gamma = \mathbf{1}_K$. Consequently (K, γ) is an injective R-module. ■

Theorem 21 *Let (E, μ) be a nonzero fuzzy R-module. The following are equivalent.*

(i) (E, μ) is injective.

(ii) $E = \mu^*$ and (E, μ) is a direct summand of every extension of itself.

Proof. (i)⇒(ii): Suppose (E, μ) is injective. Then $\mu = \mathbf{1}_E$ and so $E = \mu^*$. Let (E', β) be an extension of (E, μ). Since E is an injective R-module and $E \subseteq E'$, E is a direct summand of E', [Theorem 2.15, 5]. Hence there exists a R-submodule K of E such that $E' = E \oplus K$. Let $\gamma = \beta|_K$. Then (K, γ) is fuzzy R-submodule of (E', β). Since $\beta(x) \geq \mu(x) = 1$ for all $x \in E$, $\beta_E = \mu$. We now show that $\beta = \mu \oplus \gamma$. Let $x \in E'$. Then x has a unique representation as $x = u + v$ for some $u \in E$ and $v \in K$. First suppose that $\gamma(v) = 1$. Then $\beta(v) = 1$. Also $\beta(u) = 1$. It now follows that $\beta(x) \geq \beta(u) \wedge \beta(v) = 1$ and so $\beta(x) = 1$. Also $(\mu \oplus \gamma)(x) = \mu(u) \wedge \gamma(v) = 1$. Thus $\beta(x) = (\mu \oplus \gamma)(x)$. Now suppose $\gamma(v) \neq 1$. Then $\beta(v) \neq 1$. Since $\beta(u) = 1 > \beta(v)$, it follows that $\beta(x) = \beta(u+v) = \beta(v) = \gamma(v) = 1 \wedge \gamma(v) = \mu(u) \wedge \gamma(v) = (\mu \oplus \gamma)(x)$. Therefore $\beta(x) = (\mu \oplus \gamma)(x)$. Thus $\beta = \mu \oplus \gamma$. Hence $(E', \beta) = (E, \mu) \oplus (K, \gamma)$, i.e., (E, μ) is a direct summand of (E', β).

(ii)⇒(i): Suppose $E = \mu^*$ and (E, μ) is a direct summand of every extension of itself. Let E' be an injective extension of E. (Note that E' exists by Theorem 2.21 [5]). Now $(E', \mathbf{1}_{E'})$ is an injective fuzzy R-module and is an extension of (E, μ). Hence there exists a fuzzy R-submodule (K, γ) of $(E', \mathbf{1}_{E'})$ such that $(E', \mathbf{1}_{E'}) = (E, \mu) \oplus (K, \gamma)$. Then $E' = E \oplus K$ and $\mathbf{1}_{E'} = \mu \oplus \gamma$. Now for all $x \in E$, $1 = \mathbf{1}_{E'}(x) = (\mu \oplus \gamma)(x) = \mu(x) \wedge \gamma(0) = \mu(x)$ since $x = x + 0$ is a unique representation of x. Thus $\mu = \mathbf{1}_E$. Since E' is injective and E is a direct summand of E', E is injective. Hence (E, μ) is an injective fuzzy R-module. ■

Theorem 22 *Let (E, μ) be a nonzero fuzzy R-module such that $E = \mu^*$. Then (E, μ) is injective if and only if (E, μ) has no proper essential extension.*

Proof. Suppose (E, μ) is injective. Then $\mu = \mathbf{1}_E$. Let (B, β) be an essential extension of (E, μ). If $E \neq B$, then B is a proper essential extension of E, a contradiction since E is an injective R-module. Hence $E = B$. This implies that $\mu = \mathbf{1}_E = \beta$. Thus (E, μ) has no proper essential extension. Conversely, suppose (E, μ) has no proper essential extension. Then since $E = \mu^*$, we must have $\mu = \mathbf{1}_E$ otherwise $(E, \mathbf{1}_E)$ is a proper essential extension of (E, μ). Since

(E,μ) has no proper essential extension, it follows that E has no proper essential extension and so E is injective. Consequently, (E,μ) is injective. ■

Proposition 23 *Let (A,α) and (E,μ) be fuzzy R-modules such that (E,μ) is a maximal essential extension of (A,α). Then E is a maximal essential extension of A.*

Proof. Since $(A,\alpha)\subseteq_e (E,\mu)$, $A\subseteq_e E$. Suppose E is not a maximal essential extension of A. Then there exists a R-module E' such that $A\subseteq_e E'$ and $E\subset E'$. Define $\mu' : E'\to[0,1]$ by for all $x\in E'$

$$\mu'(x)=\begin{cases}\mu(x) & \text{if } x\in E\\ 0 & \text{if } x\notin E.\end{cases}$$

Then (E',μ') is a fuzzy R-module and $(A,\alpha)\subseteq(E',\mu')$. Also $(E,\mu)\subset(E',\mu')$. Let (K,γ) be a nonzero fuzzy R-submodule of (E',μ'). Then $A\cap K\neq\{0\}$. Now γ^* is a nonzero R-submodule of K and hence of E'. Let $x\in\gamma^*$. Then $0<\gamma(x)\leq\mu'(x)$. If $x\notin E$, then $\mu'(x)=0$, a contradiction. Hence $x\in E$. Thus $\gamma^*\subseteq E$. Also $\gamma(x)\leq\mu'(x)=\mu(x)$. Hence (γ^*,γ) is a nonzero fuzzy R-submodule of (E,μ). Thus $\alpha\cap\gamma\neq\theta$. It now follows that $(A,\alpha)\subseteq_e(E',\mu')$, a contradiction since (E,μ) is a maximal essential extension of (A,α). Hence E is a maximal essential extension of A. ■

Proposition 24 *Let (A,α) and (E,μ) be fuzzy R-modules such that $A=\alpha^*$. If E is a maximal essential extension of A, then $(E,\mathbf{1}_E)$ is a maximal essential extension of (A,α).*

Proof. Suppose E is a maximal essential extension of A. Let (K,γ) be a nonzero fuzzy R-submodule of $(E,\mathbf{1}_E)$. Then $A\cap K\neq\{0\}$. Now γ^* is a nonzero R-submodule of K and hence of E. Thus $A\cap\gamma^*\neq\{0\}$. Let $0\neq x\in A\cap\gamma^*=\alpha^*\cap\gamma^*$. This implies that $(\alpha\cap\gamma)(x)>0$. Therefore $\alpha\cap\gamma\neq\theta$. Hence $(A,\alpha)\subseteq_e(E,\mathbf{1}_E)$. Suppose $(A,\alpha)\subseteq_e(E,\mathbf{1}_E)\subseteq(F,\zeta)$ and $(A,\alpha)\subseteq_e(F,\zeta)$, where (F,ζ) is a fuzzy R-module. Then $A\subseteq_e E\subseteq F$ and $A\subseteq_e F$. Since E is a maximal essential extension of A, we must have $E=F$. Since $\zeta(x)\geq\mathbf{1}_E(x)=1$ for all $x\in E$, it follows that $\zeta=\mathbf{1}_E$. Hence $(E,\mathbf{1}_E)$ is a maximal essential extension of (A,α).
■

The following example shows that Proposition 24 need not be true if $A\neq\alpha^*$.

Example 25 *Let $\mathbb{Q}$ be the set of rational numbers and $\mathbb{Z}$ be the set of integers. Then $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}\oplus\mathbb{Z}$ and $\mathbb{Q}\oplus\mathbb{Q}$ are $\mathbb{Z}$-modules and $\mathbb{Q}_{\mathbb{Z}}$ is the injective hull of $\mathbb{Z}_{\mathbb{Z}}$. So $(\mathbb{Q}\oplus\mathbb{Q})_{\mathbb{Z}}$ is the injective hull of $(\mathbb{Z}\oplus\mathbb{Z})_{\mathbb{Z}}$. Thus $\mathbb{Q}\oplus\mathbb{Q}$ is a maximal essential extension of $\mathbb{Z}\oplus\mathbb{Z}$. Write $\mu=\mathbf{1}_{\mathbb{Q}\oplus\mathbb{Q}}$. Define $\alpha:\mathbb{Z}\oplus\mathbb{Z}\to[0,1]$ by*

$$\alpha(x)=\begin{cases}1 & \textit{if } x\in\mathbb{Z}\oplus\{0\}\\ 0 & \textit{otherwise.}\end{cases}$$

Then $(\mathbb{Z}\oplus\mathbb{Z},\,\alpha)$ is a nonzero fuzzy $\mathbb{Z}$-module of $(\mathbb{Q}\oplus\mathbb{Q},\,\mathbf{1}_{\mathbb{Q}\oplus\mathbb{Q}})$. Define $\delta:\mathbb{Q}\oplus\mathbb{Q}\to[0,1]$ by

$$\delta(x)=\begin{cases}1 & \textit{if } x\in\{0\}\oplus\mathbb{Z}\\ 0 & \textit{otherwise.}\end{cases}$$

Then $(\mathbb{Q}\oplus\mathbb{Q},\ \delta)$ *is a nonzero fuzzy* $\mathbb{Z}$*-module of* $(\mathbb{Q}\oplus\mathbb{Q}\ ,\ \mathbf{1}_{\mathbb{Q}\oplus\mathbb{Q}})$*. Let* $t\in\mathbb{Q}\oplus\mathbb{Q}$*,* $t\neq(0,0)$*. If* $t\in\mathbb{Z}\oplus\{0\}$*, then* $t\notin\{0\}\oplus\mathbb{Z}$ *and so* $\delta(t)=0$*. If* $t\notin\mathbb{Z}\oplus\{0\}$*, then* $\alpha(t)=0$*. Thus* $(\alpha\cap\delta)(t)=\alpha(t)\wedge\delta(t)=0\wedge0=0$*. Hence* $\alpha\cap\delta=\theta$*. Thus* $(\mathbb{Q}\oplus\mathbb{Q}\ ,\ \mathbf{1}_{\mathbb{Q}\oplus\mathbb{Q}})$ *is not an essential extension of* $(\mathbb{Z}\oplus\mathbb{Z},\ \alpha)$*.*

Definition 26 *Let* (A,α) *be a fuzzy* R*-module. A fuzzy* R*-module* (E,μ) *is called an* ***injective hull (injective envelope)*** *of* (A,α)*, if* (E,μ) *is injective and* $(A,\alpha)\subseteq_e(E,\mu)$*.*

Theorem 27 *Let* A *be a* R*-module such that* $A=B\oplus C$ *for some nonzero* R*-submodules* B *and* C *of* A*. Define* $\alpha:A\rightarrow[0,1]$ *by for all* $x\in A$*,*

$$\alpha(x)=\begin{cases}1 & \textit{if } x\in B\\ 0 & \textit{otherwise.}\end{cases}$$

(i) (A,α) *is a fuzzy* R*-module.*

(ii) There does not exist any injective fuzzy R*-module* (E,μ) *such that* $(A,\alpha)\subseteq_e(E,\mu)$*, i.e.,* (A,α) *does not have an injective hull.*

Proof. (i) Since α_t is an R-submodule of A for all $t\in\mathrm{Im}(\alpha)$, (A,α) is a fuzzy R-module.

(ii) Suppose there exists an injective fuzzy R-module (E,μ) such that $(A,\alpha)\subseteq_e(E,\mu)$. Then E is injective and $\mu=\mathbf{1}_E$. Define $\delta:E\rightarrow[0,1]$ by for all $x\in E$,

$$\delta(x)=\begin{cases}1 & \text{if } x\in C\\ 0 & \text{otherwise.}\end{cases}$$

Then (E,δ) is a nonzero fuzzy R-submodule of (E,μ). Since $(A,\alpha)\subseteq_e(E,\mu)$, $\alpha\cap\delta\neq\theta$. Let $x\in E$, $x\neq0$. If $x\in B$, then $x\notin C$ and so $\delta(x)=0$. If $x\notin B$, then $\alpha(x)=0$. Hence it now follows that $(\alpha\cap\delta)(x)=\alpha(x)\wedge\delta(x)=0$. Thus $\alpha\cap\delta=\theta$, a contradiction. Hence there does not exist any injective fuzzy R-module (E,μ) such that $(A,\alpha)\subseteq_e(E,\mu)$. ■

Theorem 28 *Let* (A,α) *be a fuzzy* R*-module such that* $A=\alpha^*$*. Then* (A,α) *has an injective hull, i.e., there exists an injective fuzzy* R*-module* (E,μ) *such that* $(A,\alpha)\subseteq_e(E,\mu)$*.*

Proof. Since A is an R-module, there exists an injective R-module E such that $A\subseteq_e E$. Now $(E,\mathbf{1}_E)$ is an injective fuzzy R-module. Let (K,δ) be a nonzero fuzzy R-submodule of $(E,\mathbf{1}_E)$. Then $A\cap K\neq\{0\}$. Now δ^* is a nonzero R-submodule of K and hence of E. Hence $A\cap\delta^*\neq\{0\}$. Let $x\neq0$, and $x\in A\cap\delta^*$. Now $x\in A=\alpha^*$ and so $\alpha(x)>0$. Also $x\in\delta^*$. Thus $(\alpha\cap\delta)(x)=\alpha(x)\wedge\delta(x)>0$. Hence $(A,\alpha)\subseteq_e(E,\mu)$. ■

The above theorem gives a sufficient condition for a fuzzy module to have an injective essential extension. The following example shows that the condition $A=\alpha^*$ is not necessary for a fuzzy R-module to have an injective essential extension.

Example 29 *Let $\mathbb{Q}$ be the set of rational numbers. Then $\mathbb{Q}$ is a $\mathbb{Z}$-module. It is known that $\mathbb{Q}_{\mathbb{Z}}$ is injective. Let n be a positive integer. Now $\mathbb{Z}$ is a $\mathbb{Z}$-submodule of $\mathbb{Q}$. Define $\mu : \mathbb{Z} \to [0,1]$, by*

$$\mu(x) = \begin{cases} 1 & \text{if } x \in n\mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

Then $(\mathbb{Z}, \mu)$ is a fuzzy $\mathbb{Z}$-module of $(\mathbb{Q}, 1_{\mathbb{Q}})$ and $\mu \neq 1_{\mathbb{Z}}$. Let (K, γ) be a nonzero fuzzy $\mathbb{Z}$-module of $(\mathbb{Q}, 1_{\mathbb{Q}})$. Let $\frac{p}{q}$ be a nonzero element of K. Then

$$p = \underbrace{\left(\frac{p}{q} + \cdots + \frac{p}{q}\right)}_{q \ times} \in K.$$

Hence $p \in K \cap \mathbb{Z}$. Thus $K \cap \mathbb{Z} \neq \{0\}$. Since $\gamma \neq \theta$, there exists $x \neq 0$, $x \in K$ such that $\gamma(x) \neq 0$. Let $x = \frac{s}{t}$ for some integer s and t. Then $\gamma(ns) = \gamma(ntx) \geq \gamma(x) \neq 0$. Also $\mu(ns) = 1$. Hence $\gamma \cap \mu \neq \theta$. Thus $(\mathbb{Q}, 1_{\mathbb{Q}})$ is an essential extension of $(\mathbb{Z}, \mu)$. Since $(\mathbb{Q}, 1_{\mathbb{Q}})$ is injective, $(\mathbb{Q}, 1_{\mathbb{Q}})$ is an injective essential extension of $(\mathbb{Z}, \mu)$.

Remark 30 *It is known that every R-module has an injective hull [5]. However, as shown by Theorem 27 not every fuzzy R-module has an injective hull. Theorem 28 gives a sufficient condition for a fuzzy R-module to have an injective hull.*

Theorem 31 *Let (A, α) be a fuzzy R-module such that $A = \alpha^*$. Then there exists a fuzzy R-module (E, μ) such that*

(i) (E, μ) is a maximal essential extension of (A, α) in the sense that if $(A, \alpha) \subseteq_e (B, \beta)$, then the inclusion map $i : (A, \alpha) \to (E, \mu)$ can be extended to a fuzzy R-monomorphism $(B, \beta) \to (E, \mu)$.

(ii) (E, μ) is a minimal injective extension of (A, α) in the sense that (E, μ) is injective and any fuzzy R-monomorphism from $(A, \alpha) \to (E', \mu')$ with (E', μ') injective extends to a fuzzy R-monomorphism from $(E, \mu) \to (E', \mu')$.

Proof. Since A is an R-module, there exists an R-module E such that

(a) E is a maximal essential extension of A in the sense that if $A \subseteq_e B$, then the inclusion map $i : A \to E$ can be extended to an R-monomorphism $B \to E$.

(b) E is a minimal injective extension of A in the sense that E is injective and any R-monomorphism from $A \to E'$ with E' injective extends to an R-monomorphism from $E \to E'$.

(i) By Proposition 24, $(E, \mathbf{1}_E)$ is a maximal essential extension of (A, α). Let $(A, \alpha) \subseteq_e (B, \beta)$ for some fuzzy R-module (B, β). Then $A \subseteq_e B$. Let $i : (A, \alpha) \to (E, \mu)$ be the inclusion map. Then $i : A \to E$ is the inclusion map. Since E is a maximal essential extension of A and $A \subseteq_e B$, there exists a R-monomorphism $f : B \to E$ such that $f|_A = i$. Now $\mathbf{1}_E(f(x)) = 1 \geq \beta(x)$ for all

$x \in B$. Hence $f : (B, \beta) \to (E, \mu)$ is a fuzzy R-monomorphism. Clearly f is an extension of i.

(ii) Now $(E, \mathbf{1}_E)$ is injective and $(A, \alpha) \subseteq (E, \mathbf{1}_E)$. Let $(A, \alpha) \subseteq (B, \beta) \subseteq (E, \mathbf{1}_E)$ with (B, β) injective for some fuzzy R-module. Then B is injective, $\beta = \mathbf{1}_B$ and $A \subseteq B \subseteq E$. Since E is a minimal injective extension of A, we must have $E = B$. It now follows that $(B, \beta) = (E, \mathbf{1}_E)$. Hence $(E, \mathbf{1}_E)$ is a minimal injective extension of (A, α). The second part can be proved as before. ■

Theorem 32 *Let (A, α) and (E, μ) be fuzzy R-modules such that $A = \alpha^*$. Then (E, μ) is an injective hull of (A, α) if and only if (E, μ) is a minimal injective extension of (A, α).*

Proof. Suppose (E, μ) is an injective hull of (A, α). Then (E, μ) is injective and $(A, \alpha) \subseteq_e (E, \mu)$. Thus (E, μ) is an injective extension of (A, α). Since (E, μ) is injective, E is injective and $\mu = \mathbf{1}_E$. Also $(A, \alpha) \subseteq_e (E, \mu)$ implies that $A \subseteq_e E$. This implies that E is an injective hull of A and hence E is a minimal injective extension of A. Suppose $(A, \alpha) \subseteq (B, \beta) \subseteq (E, \mu)$ with (B, β) injective for some fuzzy R-module (B, β). Then B is injective, $\beta = \mathbf{1}_B$, and $A \subseteq B \subseteq E$. Thus $B = E$. This implies that $\beta = \mu$. Hence (E, μ) is a minimal injective extension of (A, α).

Conversely, suppose (E, μ) is a minimal injective extension of (A, α). Then E is injective and $\mu = \mathbf{1}_E$. It is easy to verify that E is a minimal injective extension of A. Thus E is an injective hull of A. Hence $A \subseteq_e E$. Since $A \subseteq_e E$ and $A = \alpha^*$, it follows that $(A, \alpha) \subseteq_e (E, \mu)$. ■

Example 33 *It is known that $\mathbb{Q}_{\mathbb{Z}}$ is injective. Let n be a positive integer. Now $n\mathbb{Z}$ is a $\mathbb{Z}$-submodule of $\mathbb{Q}$. Define $\mu : \mathbb{Z} \to [0, 1]$, by*

$$\mu(x) = \begin{cases} 1 & \text{if } x \in n\mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

Then $(\mathbb{Z}, \mu)$ is a fuzzy $\mathbb{Z}$-module of $(\mathbb{Q}, 1_{\mathbb{Q}})$. Let (K, γ) be a nonzero fuzzy $\mathbb{Z}$-module of $(\mathbb{Q}, 1_{\mathbb{Q}})$. Let $\frac{p}{q}$ be a nonzero element of K. Then $p \in K \cap \mathbb{Z}$. Since $\gamma \neq \theta$, there exists a $x \neq 0$, $x \in K$ such that $\gamma(x) \neq 0$. Let $x = \frac{s}{t}$ for some integer s and t. Then $\gamma(ns) = \gamma(ntx) \geq \gamma(x) \neq 0$. Also $\mu(ns) = 1$. Hence $\gamma \cap \mu \neq \theta$. Thus $(\mathbb{Q}, 1_{\mathbb{Q}})$ is an essential extension of $(\mathbb{Z}, \mu)$. Since $(\mathbb{Q}, 1_{\mathbb{Q}})$ is injective, $(\mathbb{Q}, 1_{\mathbb{Q}})$ is the injective hull of $(\mathbb{Z}, \mu)$. In fact it can be shown that $(\mathbb{Q}, 1_{\mathbb{Q}})$ is the injective hull of $(n\mathbb{Z}, 1_{n\mathbb{Z}})$.

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